Discrete Mathematics Recitation Course 7

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7-1

Recurrence Relations

7-1 Ex.8

- Find the solution to the recurrence relation with the given initial condition.
- c) $a_n = a_{n-1} - n, a_0 = 4$

- $a_n = -n + a_{n-1}$
- $= -n + (-(n-1) + a_{n-2}) = -\big(n + (n-1)\big) + a_{n-2}$
- $= -\big(n + (n-1)\big) + (-(n-2) + a_{n-3}) = -\big(n + (n-1) + (n-2)\big) + a_{n-3}$
- $\dots$
- $= -\Big(n + (n-1) + (n-2) + \cdots + \big(n - (n-1)\big)\Big) + a_{n-n}$
- $= -(n + (n-1) + (n-2) + \cdots + 1) + a_0$
- $= -\frac{n(n+1)}{2} + 4 = \frac{-n^2-n+8}{2}$

7-1 Ex.24

- a) Find a recurrence relation for the number of bit strings of length n that contain three consecutive 0s.
- b) What are the initial conditions?
- c) How many bit strings of length seven contain three consecutive 0s?

- X1 => a_{n-1} , X10 => a_{n-2} , X100 => a_{n-3} ,
- X000 => 2^{n-3}
- for $n \geq 3, a_n = a_{n-1} + a_{n-2} + a_{n-3} + 2^{n-3}$
- $a_0 = a_1 = a_2 = 0$
- 47

7-1 Ex.30

- a) Find a recurrence relation for the number of ternary strings of length n that contain two consecutive 0s.
- b) What are the initial conditions?
- c) How many ternary strings of length six contain two consecutive 0s?

- X1 => a_{n-1} , X2 => a_{n-1} , X10 => a_{n-2} , X20 => a_{n-2}
- X00 => 3^{n-2}
- for $n \geq 2, a_n = 2a_{n-1} + 2a_{n-2} + 3^{n-2}$
- $a_0 = a_1 = 0$
- 281

7-2

Solving Linear Recurrence Relations

7-2 Ex.4

- Solve these recurrence relations together with the initial conditions given
- a) $a_n = a_{n-1} + 6a_{n-2}$ for $n \geq 2, a_0 = 3, a_1 = 6.$
- f) $a_n = -6a_{n-1} - 9a_{n-2}$ for $n \geq 2, a_0 = 3, a_1 = -3.$

- $r^2 - r - 6 = 0,\ r = -2, 3$
- $a_n = \alpha_1(-2)^n + \alpha_2 3^n$
- $3 = \alpha_1 + \alpha_2$
- $6 = -2\alpha_1 + 3\alpha_2$
- $\alpha_1 = \frac{3}{5},\ \ \alpha_2 = \frac{12}{5}$
- $a_n = (3/5)(-2)^n + (12/5)3^n$

7-2 Ex.4 (cont'd)

- $r^2 + 6r + 9 = 0 \quad r = -3, -3$
- $a_n = \alpha_1(-3)^n + \alpha_2 n(-3)^n$

- $3 = \alpha_1$
- $-3 = -3\alpha_1 - 3\alpha_2$
- $\alpha_1 = 3 \quad \alpha_2 = -2$

- $a_n = 3(-3)^n - 2n(-3)^n = (3 - 2n)(-3)^n$

7-2 Ex.12

- Find the solution to $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$ with $a_0 = 3, a_1 = 6, a_2 = 0.$

- $r^3 - 2r^2 - r + 2 = 0,\ r = 1,\ -1, 2$
- $a_n = \alpha_1 + \alpha_2(-1)^n + \alpha_3 2^n$
- $3 = \alpha_1 + \alpha_2 + \alpha_3$
- $6 = \alpha_1 - \alpha_2 + 2\alpha_3$
- $0 = \alpha_1 + \alpha_2 + 4\alpha_3$

- $a_n = 6 - 2(-1)^n - 2^n$

7-2 Ex.18

- Solve the recurrence relation $a_n = 6a_{n-1} - 12a_{n-2} + 8a_{n-3}$ with $a_0 = -5, a_1 = 4, a_2 = 88$.

- $r^3 - 6r^2 + 12r - 8 = 0 \equiv (r-2)^3, r = 2, 2, 2$
- $a_n = \alpha_1 2^n + \alpha_2 n 2^n + \alpha_3 n^2 2^n$
- $-5 = \alpha_1$
- $4 = 2\alpha_1 + 2\alpha_2 + 2\alpha_3$
- $88 = 4\alpha_1 + 8\alpha_2 + 16\alpha_3$

- $a_n = -5 \cdot 2^n + \frac{1}{2} \cdot n2^n + \frac{13}{2} n^2 2^n$
  $= -5 \cdot 2^n + (n + 13n^2) \cdot 2^{n-1}$

7-2 Ex.20

- Find the general form of the solutions of the recurrence relation $a_n = 8a_{n-2} - 16a_{n-4}$.

- $r^4 - 8r^2 + 16 = 0 \equiv (r^2 - 4)^2 \equiv (r-2)^2(r+2)^2$
- $r = 2, 2, -2, -2$
- $a_n = \alpha_1 2^n + \alpha_2 n 2^n + \alpha_3(-2)^n + \alpha_4 n(-2)^n$

7-2 Ex.22

- What is the general form of the solutions of a linear homogeneous recurrence relation if its characteristic equation has the roots $-1, \ -1, \ -1, \ 2, \ 2, \ 5, \ 5, \ 7$?

- $a_n = (\alpha_{1,0} + \alpha_{1,1}n + \alpha_{1,2}n^2)(-1)^n + (\alpha_{2,0} + \alpha_{2,1}n)2^n + (\alpha_{3,0} + \alpha_{3,1}n)5^n + \alpha_{4,0}7^n$

7-2 Ex.24

- Consider the nonhomogeneous linear recurrence relation $a_n = 2a_{n-1} + 2^n$.
  - a) Show that $a_n = n2^n$ is a solution.
  - b) Find all solutions of this recurrence relation.
  - c) Find the solution with $a_0 = 2$.

- $2(n-1)2^{n-1} + 2^n = (n-1)2^n + 2^n = n2^n$
- ${a_n}^{(h)}$ is $a_n = \alpha 2^n$, so $a_n = \alpha 2^n + n2^n$
- $2 = \alpha$, $a_n = 2 \cdot 2^n + n2^n = (2+n)2^n$

7-2 Ex.26

- What is the general form of the particular solution guaranteed to exist by the theorem of the linear nonhomogeneous recurrence relation $a_n = 6a_{n-1} - 12a_{n-2} + 8a_{n-3} + F(n)$ if
- a) $F(n) = n^2$? $p_2n^2 + p_1n + p_0$
- b) $F(n) = 2^n$? $n^3p_02^n$
- c) $F(n) = n2^n$? $n^3(p_1n + p_0)\ 2^n$
- d) $F(n) = (-2)^n$? $p_0(-2)^n$
- e) $F(n) = n^22^n$? $n^3(p_2n^2 + p_1n + p_0)\ 2^n$
- f) $F(n) = n^3(-2)^n$? $(p_3n^3 + p_2n^2 + p_1n + p_0)\ (-2)^n$
- g) $F(n) = 3$? p_0

7-3

Divide-and-Conquer Algorithms and Recurrence Relations

7-3 Ex.8

- Find $f(n)$ when $n = 3^k$, where f satisfies the recurrence relation $f(n) = 2f(n/3) + 4$ with $f(1) = 1$.
- $f(n) = C_1 n^{\log_b a} + C_2$ where
- $C_1 = \frac{c}{a-1} + f(1)$ and $C_2 = \frac{-c}{a-1}$
- Now $a = 2$, $b = 3$, $c = 4$, and $f(1) = 1$
- $C_1 = 5$ and $C_2 = -4$
- $f(n) = 5n^{\log_3 2} - 4$

7-3 Ex.9

- Suppose that there are $n = 2^k$ teams in an elimination tournament, where there are $n/2$ games in the first round, with the $n/2 = 2^{k-1}$ winners playing in the second round, and so on. Develop a recurrence relation for the number of rounds in the tournament. Also, Solve the recurrence relation for the number of rounds in the tournament.

7-3 Ex.9 (cont.)

- If there is only one team, then no rounds are needed, so the base case is $R(1) = 0$. Since it takes one round to cut the number of teams in half, we have $R(n) = 1 + R(n/2)$
- $R\left(2^k\right) = a^k R(1) + c\sum_{i=0}^{k-1} a^i$
- $c = 1$, $a = 1$, and $R(1) = 0$.
- $R\left(2^k\right) = \sum_{i=0}^{k-1} 1^i = k$

Master theorem examples

Ex. 1: $T(n) = 4T(\frac{n}{2}) + n$

- Compare $f(n) = n$ with $n^{\log_b a} = n^2$: $f(n) = n = O(n^{2-\epsilon})$.
- Case 1 applies: $T(n) = \Theta(n^{\log_b a}) = \Theta(n^2)$.

Ex. 2: $T(n) = 2T(\frac{n}{2}) + n$

- Compare $f(n) = n$ with $n^{\log_b a} = n$: $f(n) = n = \Theta(n)$.
- Case 2 applies: $T(n) = \Theta(n^{\log_b a} \lg n) = \Theta(n \lg n)$.

Master theorem examples

Ex. 3: $T(n) = 3T(\frac{n}{4}) + n \lg n$

- Compare $f(n) = n \lg n$ with $n^{\log_b a} = n^{0.79}$: $f(n) = n \lg n = \Omega(n^{0.79+\epsilon})$.
- Case 3 **could** apply: Need to **check for "regularity" condition** that $af(n/b) \le cf(n)$.
  * Find $c < 1$ s.t. $af(n/b) \le cf(n)$ for large enough n.
  * $3\frac{n}{4} \lg \frac{n}{4} \le cn \lg n$ which is true for $c = \frac{3}{4}$.
- Case 3 applies: $T(n) = \Theta(f(n)) = \Theta(n \lg n)$.

7-5

Inclusion-Exclusion

7-5 Ex.6

- In a survey of 270 college students, it is found that 64 like coke, 94 like red tea, 58 like green tea, 26 like both coke and red tea, 28 like both coke and green tea, 22 like both red tea and green tea, and 14 like all of them. How many of the 270 students do not like any of these beverages.

- 270-64-94-58+26+28+22-14=116